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Professor Cayley, to whom I communicated the results of §§ 16–18 some time ago was, I found, already acquainted with them; but it seems possible that they may not have been published before, as it would not be easy to express them without the aid of determinants, and the only paper I have met with in which an explicit use of determinants is made in connection with the solution of equations in the method of least squares is that of Mr. Van Geer referred to in the note in § 14.

On the Possible Performance of an Object-Glass for Star-Gazing.

By Edward Sang, Esq.

It having been proposed to compute the curvatures for an object-glass, with a view to obtaining the least possible aberration in the image of a star, the preliminary question arose as to how the computations should be conducted.

In the preparation of formulæ for the amount of spherical aberration, the sines and cosines of arcs are represented by two or three terms of the series which truly express them, and therefore such formulæ are only approximative, and the results obtained by them are to be regarded as guides to more accurate determinations. Our ultimate resort is to trace strictly the course of each pencil of light. When the thicknesses are taken into account, the application of the formulæ becomes as laborious as the direct trigonometrical calculation itself; wherefore it was determined to follow the trigonometrical method throughout.

The computation thus takes the form of a series of trials applicable only to the particular case in hand, and we have so to arrange these trials as to make them exhaustive, and so also as to throw light on analogous cases.

The proposition as it occurs in practice is this:—" Given two discs of glass, to construct of them an object-glass which shall give the best possible result." In the present instance that result is to be the formation of the image of an exceedingly minute luminous object; the correlative matters of the flatness of the field of view and of the performance towards the edge of that field not being taken into account. Now, the refractions by the two kinds of glass are data in the problem and fix the amount of the secondary chromatic aberration; wherefore our enquiry must be mainly directed to that part of the total error, which depends on the sphericity of the surfaces—that being the only matter under the control of the constructor.

The case actually proposed was to make an aplanatic combination from two discs, one of *hard-crown*, the other of *dense flint* glass, having an aperture of 7·5 inches, with a thickness in the rough of ·75; the indices of refraction being given in Chance's list as under, and the focal distance to be about 100 inches.

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Glass (Chance's).	Density.	C	D	F	G
Hard Crown	2·485	1·5146	1·5172	1·5232	1·5280
Soft Crown	2·55	1·5119	1·5146	1·5210	1·5263
Light Flint	3·21	1·5700	1·5740	1·5839	1·5922
Dense Flint	3·66	1·6175	1·6224	1·6348	1·6453
Extra Dense	3·85	1·6450	1·6504	1·6643	1·6761

The investigation was divided into two parts: one in which the extreme bands C and G alone were considered; the other, in which all the four were taken into account.

The first step was to compute, as for an absolutely thin lens, the curvatures needed to bring together the images for the exreme bands C and G, the result being:—

$$\frac{1}{r_1} - \frac{1}{r_2} = -·0460\ 9225,$$

$$\frac{1}{r_3} - \frac{1}{r_4} = +·0222\ 1714\ ;$$

so that the thickness of the crown-glass lens may conveniently be fixed at ·70, that of the flint-glass one at ·54, measured on the axis of the system.

These equations show no connection between the second and third curvatures; the internal surfaces may be made coincident, or the glasses may have a lens of air between them, which may be either convex or concave; hence our trials must be arranged in sets, according to the arbitrarily assumed character of this intermediate lens.

Having established the relation between the second and first curvatures, indicated by the equation

$$\frac{1}{r_2} = \frac{1}{r_1} + ·0460\ 9225,$$

and that between the third and second curvatures by

$$\frac{1}{r_3} = \frac{1}{r_2} + \text{arbitrary constant},$$

the fourth curvature has to be computed so as to make the foci for the bands C and G coincident for those pencils of light which pass close to the axis; this computation is so simple as to need no remark. The influence of the thicknesses of the lenses is seen in the resulting curvature and position of the focus. In this way the assumption of the first curvature determines the whole arrangement in any one of the sets of trials.

It remains for us to compute the aberrations for those pencils of light which pass at the margin of the lens.

This computation involves only the very simplest cases of the resolution of triangles, and would need no special notice but for the circumstance that, in consequence of the angles being small,

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the interpolation of the arc from its logarithmic sine, or contrariwise, is laborious. With Taylor's table to each second of arc, the labour is very great. In the reprint of Taylor's given in Shortrede's Tables, I caused to be inserted tables of proportional parts, but even these fail to give us help, because for our purpose second and even third differences are needed. To meet the requirements of the case, I changed entirely the plan of operation, and, measuring the arcs by the same unit as their sines, constructed a canon of logarithmic sines for each ten-thousandth part of the radius up to ·3000. In this way the interpolations were rendered incomparably more easy, and the accuracy was correspondingly augmented so that the results have all the precision that is attainable by the use of seven-place tables.

It was natural to begin our trials with the second and third surfaces coincident. Assuming then $r_2 = r_3$ and taking for $\frac{1}{r_1}$ values ranging from $+·005$ to $-·045$, the distances of the foci from the fourth surface were computed for the achromatised central pencil, and for the marginal pencils corresponding to the bands C and G. The results are shown in the accompanying figure 1, enlarged to make the parts visible.

The numbers on the horizontal scale show the focal distances in inches (enlarged two times); those on the vertical scale, the curvatures, $\frac{1}{r_1}$, of the first surface, + indicating concavity toward the object, − indicating convexity.

The vertical line at 100 shows the position of the achromatised focus for an absolutely thin lens; the strong curved line with its convexity to the left shows the focus for the central achromatised ray of the actual lens, and the two lines convex to the right indicate the foci for the marginal pencils of the bands C and G. These are well apart for the case

$$\frac{1}{r_1} = +·005;$$

they become closer and cross each other about

$$\frac{1}{r_1} = -·040,$$

but their separation is too small to be shown even on the enlarged scale. The horizontal distances between one of these curves and the curve for the central pencil give the respective amounts of longitudinal aberration, and the points of crossing show the conditions for aplanatism for the particular band. Thus the spherical aberration of the band G is *nil* when

$$\frac{1}{r_1} = -·0065,$$

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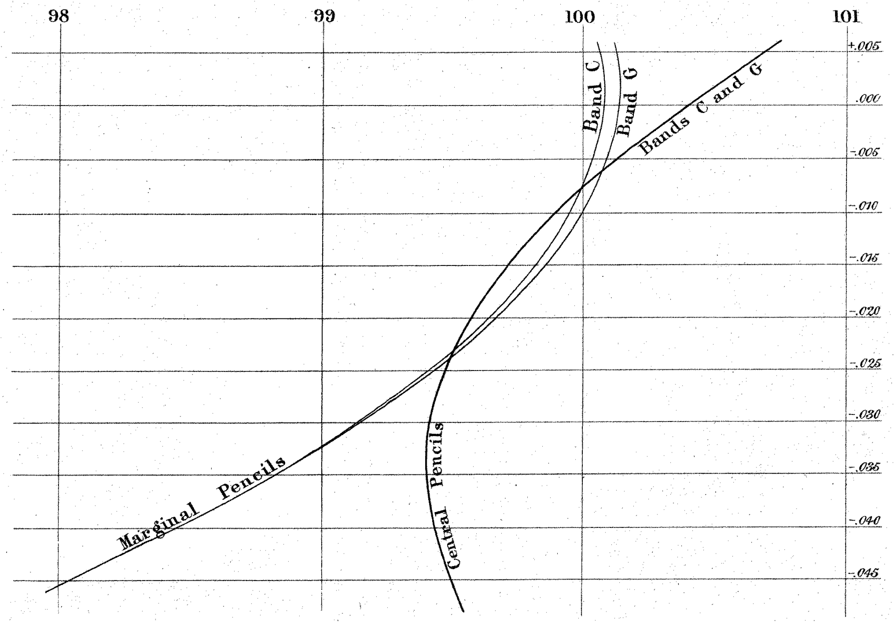
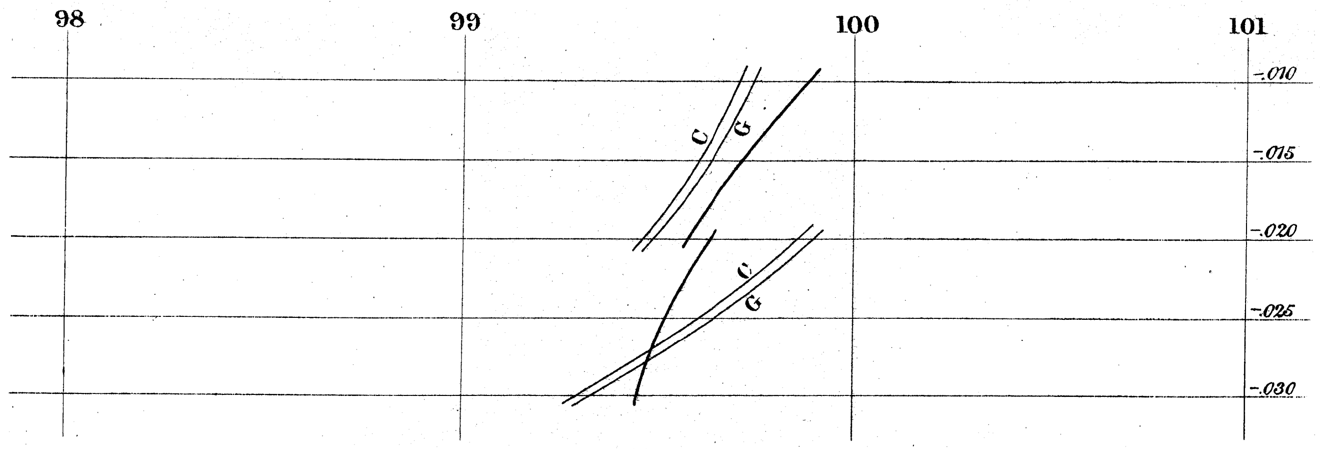
Figure 1.



Spottiswoode & C° Lith. London.

Figure 2.



Spottiswoode & C^o. Lith. London.

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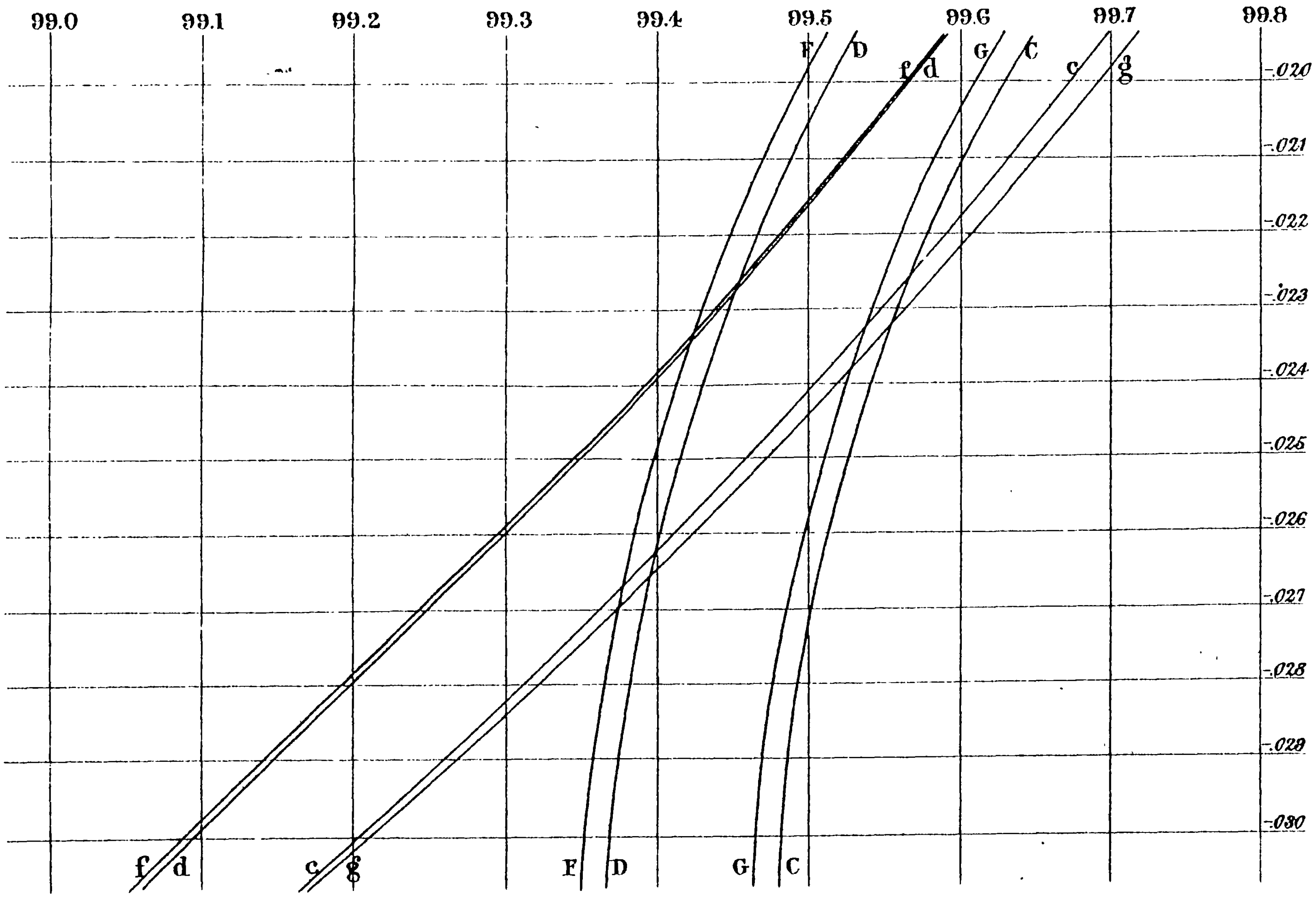
Figure 4.



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that for band C when

$$\frac{1}{r_1} = -\cdot 0078.$$

The marginal line for C again crosses at $-\cdot 0230$, that for G at $-\cdot 0238$: so that our problem has two approximate solutions, neither of them satisfactory; the separation of the foci for the marginal pencils C and G being in the latter solution ·0296 or about $\frac{1}{34}$th of an inch.

Having failed to obtain complete aplanatism with the interior surfaces coincident, we proceed to try the effect of a separation. Assuming as the basis of our second set of trials

$$\frac{1}{r_2} = \frac{1}{r_2} + \cdot 001,$$

and making the lenses touch at the edge, we get a separation of ·007 at the centre, and allow for this in our calculations; it is not necessary to carry these beyond the limits

$$\frac{1}{r_1} = -\cdot 020 \quad \text{and} \quad \frac{1}{r_1} = -\cdot 030.$$

The results are shown in the lower part of the second figure. Here the line for the central achromatised pencil is slightly, while the lines for the marginal pencils are considerably removed from the lens, so that if the curves were continued, the crossings would be much more apart than in the first figure. The separation, also, of the foci for the marginal pencils C and G is somewhat greater than before, and it is needless to proceed further in this direction.

On reversing the mode of variation by supposing

$$\frac{1}{r_3} = \frac{1}{r_2} - \cdot 001,$$

the lenses are made to touch at the centre, and their edges would need to be kept apart by means of a thin ring. The results are shown in the upper part of the second figure and are contained within the limits

$$\frac{1}{r_1} = -\cdot 010 \quad \text{and} \quad \frac{1}{r_1} = -\cdot 020.$$

The central line is here nearer to the lens; the marginal lines, however, much more so, so much that they do not cross the central line at all, and thus, under this supposition, we cannot correct the spherical aberration. For the purpose of comparing

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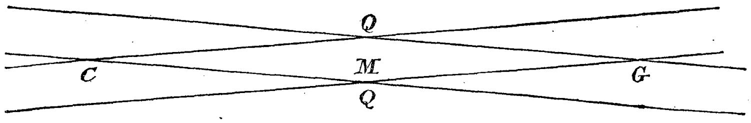
these results the second figure is made in the same form as the first one. The contrast shows us that for these two kinds of glass the best result is to be got (best in so far as the bands C and G are concerned) by making the curvature of the first surface about $-\cdot0238$, with the second and third surfaces coincident; in which case the fourth surface is sensibly flat.

FIG. 3.



If the curvatures be so adjusted as that the focus for the central achromatised pencil be midway between the foci for the marginal pencils C and G, as shown in figure 3, and if from C and G we draw lines to the edge of the lens, which lines have, in this case, an inclination of ·0375 to the axis, the least section of the compound cones thus formed will be on the focal plane passing through M. Now, the distance C G is ·0296; wherefore the radius M Q of this least section is

$$\cdot0148 \times \cdot0375 = \cdot000555:$$

that is, its diameter will be about the 900th part of an inch, in our case. This subtends, at the distance of 100 inches, an angle of $2''\cdot29$ of the ancient division of the quadrant.

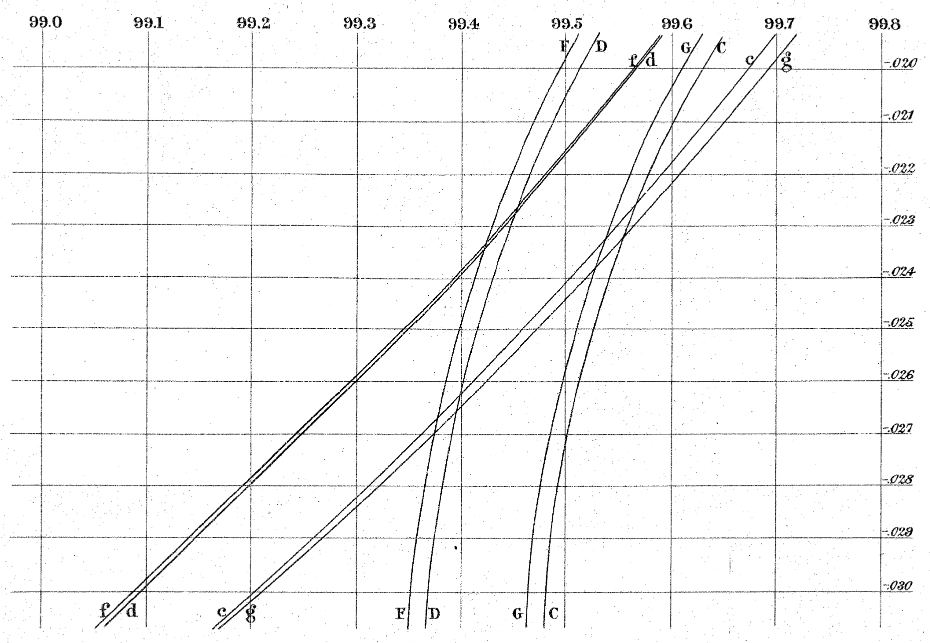
If, instead of making the central foci C and G to agree, we arrange the curvatures so as to place G before C to the extent ·0148, the central foci for C and G will agree, alternately, with the marginal foci for G and C, and the diameter of the above false disc will be halved. Thus we see that, when the aperture of the lens is ·075 of its focal length, the best performance attainable with these two kinds of glass and for the bands C and G, gives a false disc $1''\cdot145$ in diameter.

Hitherto we have considered only the extreme bands C and G; the bringing of these together does not necessarily produce achromatism, nor, indeed, can we expect that a better result might not have been reached by so treating some other pair. In combining more than two bands we cannot bring all the foci into one, and the best effect is to be got by making the sum of the squares of their mutual distances as small as possible. Also, if we desire to take into consideration the breadths and intensities of the parts of the spectrum represented by the various bands, we must attach weight-coefficients to them, so that if C represent the focus, and c the intensity of the band C, and similarly of the others, the sum

$$cd\,.\,\mathrm{CD}^2+cf\,.\,\mathrm{CF}^2+cg\,.\,\mathrm{CG}^2+df\,.\,\mathrm{DF}^2+dg\,.\,\mathrm{DG}^2+fg\quad\mathrm{FG}^2$$

is to be made a minimum.

Figure 4.



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There is no difficulty in making this calculation for the central pencil. For our present purpose it may suffice to regard the four bands as of equal intensity; and thus we may substitute for the condition $CG^2=0$ this improved one

$$CD^2+CF^2+CG^2+DF^2+DG^2+FG^2 = \text{minimum}.$$

Proceeding exactly as before, we obtain for the conditions of optimism in a thin lens, the equations

$$\frac{1}{r_1}-\frac{1}{r_2} = -\cdot0459\ 2642,$$

$$\frac{1}{r_2}-\frac{1}{r_4} = +\cdot0220\ 8918,$$

not differing much from those formerly used.

Using the first of these equations with the same thicknesses, ·70 and ·54, as before; making the interior surfaces coincident, and computing the fourth surface so as to produce the optimum effect for the central pencils; and thereafter calculating the positions of the foci for the marginal pencils, we obtain the results exhibited in figure 4. Here the longitudinal scale has been magnified eight times: that is, each inch in the drawing represents one-eighth part of an inch in focal length. Of the eight lines shown in the figure, the four marked with capital letters indicate the foci for the central pencils of the several bands; those marked in small letters serve for the marginal pencils.

A glance at this diagram shows that the best effect is to be got with a front curvature between −·02315, where the line *gg* intersects CC, and −·02330 where *ff* cuts FF. The interval between the foci for the marginal pencils of the bands F and G is then ·1259, so that the diameter of the false disc is ·004721 which subtends at the lens an angle of 9″·8—say ten seconds.

We might now proceed to make a difference between the curvatures of the second and third surfaces, but seeing that the chromatic aberration much exceeds the error from sphericity, and seeing how closely the present resembles the former case, we can hardly expect any improvement, and thus, for these two kinds of glass, we may hold that the diameter of the false image of a minute point cannot be made less than ten seconds.

Although these enquiries have had special reference to one particular combination of glasses, they lead to some general conclusions. The residual error is seen to arise from two sources: the first and the most important being in the dispersive powers, the second in the sphericity of the surfaces of the lenses; which second is only less important than the other because it is, to a certain degree, under our control.

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It is, then, of the utmost importance to select from among the different kinds of glass that are available that pair which gives the greatest concentration of the foci; and our endeavours towards the improvement of the refracting telescope should be mainly directed to the dispersive powers of the glasses. If, in our first example, the indices for the bands D and F had been such as to bring all the four foci for the central pencil together, the false disc would have only been 1″·14, or about one-eighth part of what, in the actual state of matters, is possible.

The importance of a proper choice among the kinds of glass may be seen by comparing the results from different combinations of the crown and flint glasses given in Chance's list. The following table shows, for an achromatic thin lens of 100 inches in focal distance, the mean longitudinal aberration from the mean focus, for the six combinations.

Mean Longitudinal Dispersion from Mean Focus.

	Hard Crown.	Soft Crown.
Light Flint	·05782	·03296
Dense Flint	·05712	·03868
Extra Dense	·05828	·04204

In the general problem "to obtain the best result from two given discs of glass," the focal length has to be considered. On computing, in several cases, the spherical aberration for pencils intermediate between the centre and the margin of the lens, these were found—as, indeed, was to have been expected—to be almost exactly proportional to the square of the distance of the incident ray from the centre. If, then, keeping the aperture unchanged, we augment the focal distance in any ratio, we reduce the longitudinal aberration in the same ratio; with twice the focal length the longitudinal spherical aberration is halved; but the inclination to the axis is also halved, wherefore the lateral aberration is reduced four times; consequently, since the distance is doubled, the angle subtended by the false disc is reduced to one-eighth part: that is, in the triplicate ratio of the focal length. So much for the spherical aberration.

On augmenting the focal length, the longitudinal chromatic aberration is augmented in the same ratio; wherefore the lateral aberration is unchanged, and the angle subtended by it is reduced in the ratio of the focal length; hence in this respect also it is advantageous to have a long telescope.

But every bit of glass is liable to internal irregularities, which the manufacturer takes many precautions to remove or to palliate. A ray of light in proceeding through the lens suffers deflection at each change of consistency, and the total effect is an indistinctness or blur in the image. This blur preserves the same angular extent whatever may be the focal length.

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Hence the total imperfection of the image consists of three parts: one, depending on the quality of the glass, constant; a second, proportional inversely to the focal length and arising from the distribution of the spectra; and a third, the spherical aberration, inversely proportional to the cube of the length.

If we make the length so great that the second and third imperfections are less than the constant one, we have gained all that is possible; thus we see that the exertions of the glass-maker to produce metal of uniform composition enable us, when we are in search of the best possible result, not to shorten but to lengthen the tube.

The investigation into the action of a telescope does not stop at the focus of the principal lens. Properly speaking, no image is formed there; the rays of light go onwards through the eye-piece, and it is only on the retina that there is an image. The action of the eye-lenses is quite as important as that of the object-glass, and it may be possible, by a proper arrangement of the curvatures, to remedy the unavoidable defects already noticed; the eye-piece should be suited specifically to the objective, and the whole instrument has to be treated as an integral system. The second part of the general subject may be treated in a future essay.

Before venturing critically to interpret the appearance of any object as seen through a refractor, the observer would need to know the character of the imperfections of the image. The false disc may be bright in the middle, fading to become imperceptible at the edge, or it may exhibit a coloured fringe; and in all cases the imperfection causes an overlapping of the images of two near objects, giving rise now to the seeming projection of a star upon the moon's disc, then to the phenomenon of Bailey's beads.

The light of an exceedingly faint object diffused over an appreciable disc by one lens may be so faint as to be unperceived, while by another lens, more happily arranged, the brightness may be enough to make the object visible.

An exact knowledge of the indices of the two glasses, and of their curvatures, would enable us to estimate the amount and character of the diffusion; hence the importance of having, along with all lenses of any magnitude, a prism of glass from the same melting, or, better, one cut from a corner which had been left projecting on the rough disc.

6 *Molendo Terrace, Edinburgh,*
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